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A note on the center of mass in non-commutative theories

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ABSTRACT: The dynamics of a stack of N D-branes is described by $U(N)$ gauge theory of which the central $U(1)$ describes the center of mass motion and the remaining $SU(N)$ describes the internal dynamics. In the non-commutative situation, these two parts are coupled by the $*$-commutator interaction. We describe here how to identify the correct decoupled U(1) center of mass subsector of $U(N)$ gauge theory for the case of the non-commutative torus. The internal dynamics remainder is *not* a theory of $SU(N)$ valued fields but has a simple description in momentum space.

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1. Introduction

The low energy theory of a stack of N D-branes is simply a $U(N)$ gauge theory with a number of scalar fields X^i in the adjoint representation. The diagonal elements of these scalar field matrices are usually interpreted as giving the position of the individual branes of the stack in the transversal direction while the off-diagonal elements arise from strings stretching between different branes.

For a flat target space, the set-up is translationally invariant in the transversal directions. Thus, by conservation of momentum, it should be possible to separate off the free centre of mass motion of the system from the internal dynamics. This is realized by observing that the central $U(1) \subset U(N)$ drops out of all commutators that make up the interaction and indeed the centre of mass coordinates $\frac{1}{N}\mathrm{tr}X^i$ are free. The remaining internal dynamics is then given by a $SU(N)$ gauge theory.

In the presence of background fields, the world-volume of the D-branes is known to become a non-commutative space[1, 2, 3, 4]. This results in the commutators being replaced by $*$-commutators

$$[f,g]_* = f * g - g * f = \frac{1}{2}(f^a * g^b + g^b * f^a)[T^a,T^b] + \frac{1}{2}(f^a * g^b - g^b * f^a)\{T^a,T^b\}$$

in terms of a basis of the representation of the gauge group. The second term however poses a problem as the anti-commutator of two representation matrices in general is not a representation matrix. Especially in the case of $SU(N)$, the anti-commutator in general is not trace-less and thus the internal dynamics $SU(N)$ seems to mix with the centre of mass motion given by the trace[5].

There is a similar problem of defining non-commutative gauge theories for other groups that are not just products of $U(N_i)$ factors. In those cases, the problem however is not as pressing as it is not in conflict with physical problems as the decoupling of the centre of mass motion. The later follows from translation invariance (note that the background fields inducing the non-commutativity are tangent to the branes while we are discussing transversal translations which are not broken by the

presence of the background fields): Especially the brane realization of SO and Sp gauge theories requires orientifold branes that project out the Kalb-Ramond field responsible for the non-commutativity and exceptional gauge groups are even more difficult to obtain from brane physics in smooth targets.

In the past, there have been several attempts to solve this problem at hand: Notably, there is [6] using enveloping algebras and the Seiberg-Witten map. This treatment is quite formal and it remains to be shown if the problem does not reappear once the enveloping algebra is represented on the fields. Furthermore, doing perturbation theory in the non-commutativity parameter bears the possibility of treating a non-local theory as a local one thereby obscuring important global properties (for example the crucial relation $U^q = V^q = 1$ below is invisible in finite order perturbation theory in θ).

The case of SO and Sp non-commutative gauge theories is also considered in [7]. That treatment however relies on a $B_{\mu\nu}$-field that is discontinuous at the location of the D-branes. Finally, there is also [8] which uses open Wilson lines to define a non-commutative $SU(N)$ gauge theory.

2. The non-commutative $U(1)_{CM}$

As a B-field in the macroscopic directions would violate Lorentz invariance, we will only consider non-commutativity in the compact directions. For concreteness, we will discuss a stack of N D2-branes that wrap a non-commutative torus T_θ. We will use the same notation as [9] and as explained there, we will make again use of the fact that by compactification we can integrate the dimension-full two-form B to a dimensionless parameter θ.

Under T-duality in one direction of the torus, the D2-branes becomes flat D1-branes that wrap the torus. In order to avoid the marginal case of D1's that ergodically wrap the volume of the T-dual torus, we further restrict ourselves to the case of rational $\theta = p/q$. This corresponds to D1's that wraps p times one cycle of the torus while wrapping the other cycle q times.

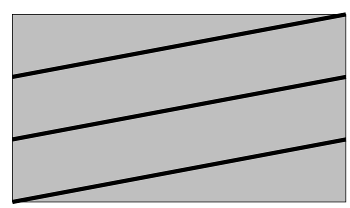

Figure 1: D1-brane dual to $\theta = 1/3$

Next, we expand all fields into their Fourier modes (we generically write $f, g, \ldots$ for fields like A_μ and X^i in the adjoint of $U(N)$ and suppress all commutative coordinates):

$$f(x_1, x_2) = \sum_{m_1, m_2} f_{m_1,m_2} e^{2\pi i m_1 x_1} e^{2\pi i m_2 x_2} = \sum_{m_1, m_2} f_{m_1,m_2} U^{m_1} V^{m_2},$$

where we introduced $U = \exp(2\pi i x_1)$ and $V = \exp(2\pi i x_2)$. The modes f_{m_1,m_2} take values in the Lie algebra of $U(N)$. The non-commutativity is summarized by

$$UV = e^{-2\pi i p/q} VU.$$

Our ordering convention is that all U's are left of the V's. The $*$-product thus reads

$$(f*g)_{m_1,m_2}=\sum_{n_1,n_2} f_{n_1,(m_2-n_2)}g_{(m_1-n_1),n_2}e^{2\pi i(m_2-n_2)(m_1-n_1)p/q}.$$

Furthermore, it is useful to split all the mode labels modulo q as $m_1=qM_1+\mu_1$ and similar with the understanding that Greek labels take values in $\mathbb{Z}_q$ as the phase factor only depends on the later:

$$(f*g)_{m_1,m_2}=\sum_{N_1,\nu_1,N_2,\nu_2} f_{N_1q+\nu_1,(M_2q+\mu_2-N_2q-\nu_2)}g_{(M_1q+\mu_1-N_1q-\nu_1),N_2q+\nu_2}e^{2\pi i(\mu_2-\nu_2)(\mu_1-\nu_1)p/q}.$$

Now our aim is to split all the fields f into a free (abelian, commutative) $\tilde{f}$ that describes the center of mass motion and a decoupled $\hat{f}=f-\tilde{f}$ that describes the internal dynamics. In the commutative situation, this is simply achieved by defining $\tilde{f}$ to be the $U(1)$ part $\frac{1}{N}\mathrm{tr}(f)$ of f while the traceless $\hat{f}$ is in the remaining $SU(N)$. As explained in the introduction, this is not sufficient in the non-commutative situation as the commutator interaction mixes these two parts.

To find the appropriate decoupled $U(1)$, it is essential to realise that the elements U^q and V^q are central, that is, they commute with all functions on the non-commutative torus. Thus we are led to define $\tilde{f}$ to contain the traces of the $(\mu_1,\mu_2)=(0,0)$ parts only:

$$\tilde{f}(x_1,x_2)=\sum_{M_1,M_2}\mathbb{I}\frac{1}{N}\mathrm{tr} f_{qM_1,qM2}U^{qM_1}V^{qM_2},$$

where $\mathbb{I}$ is the unit matrix in the Lie algebra of $U(N)$. As above, we take $\hat{f}=f-\tilde{f}$. Then it is easy to check that indeed $\tilde{f}$ decouples, that is

$$[\tilde{f},g]_*=0 \qquad \text{and} \qquad \widetilde{[f,g]_*}=0. \tag{2.1}$$

The theory of the $\tilde{f}$'s is indeed a commutative $U(1)$ gauge theory that we interpret as to describe the centre of mass motion of the stack of branes.

Note that our definition is non-local in the following sense: Instead of taking $SU(N)$ and $U(1)$ valued matrices over each point we only generalise $SU(N)$- and $U(1)$-valued functions to the non-commutative case by making use of the Fourier decomposition. Therefore we modify the gauge group rather than the structure group. But this is in the spirit of non-commutative geometry where attention is shifted from individual points to the algebra of functions.

Furthermore, the second equation in (2.1) suggests the generalization of $SU(N)$-valued functions to more general non-commutative spaces (starting with the torus with irrational θ): These are the functions that can be written as linear combinations of $*$-commutators of $U(N)$-valued functions[1]. There are, however, thorny analytical

[1] We thank P. Schupp for suggesting this

problems to be solved: Probably, one should consider the closure of the image of the commutator (e.g. allow infinite, converging sums) but the case of the irrational torus shows that one must not include too many limit points if one does not want the fine, ergodic structure of that theory. We leave this for future investigation. Similarly, we hope to gain better understanding of the relation of our definition of $\hat{f}$ with the approach of [6], that defines the non-commutative $SU(N)$ theory in terms of the image of the Seiberg-Witten map of its commutative counter-part. A counting of degrees of freedom however shows that this relation, if existent, has to be non-local.

Let us close this note with two related comments: First, as explained in [9], it is a manifestation of Morita equivalence that the algebra of U and V can be realized in terms of clock an shift matrices in $U(q)$. In that description, we are dealing with a $U(N) \otimes U(q)$ gauge theory on a commutative torus with both radii smaller by a factor q. Our split $f = \tilde{f} + \hat{f}$ there corresponds to the split of that gauge group into $U(1)$ and $S(U(N) \otimes U(q))$. This interpretation also implies that the internal theory of the $\hat{f}$ can be defined as a quantum theory as it can be rewritten as an ordinary commutative gauge theory.

Second, the fact that only every qth Fourier mode appears in $\tilde{f}$ can be motivated as well in the T-dual description: There the wrapping stack of D1-branes looks like q D1-stacks. Thus only every qth mode describes the collective motion of the q stacks which is the center of mass motion, see the figure.

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